Local-Effect Games

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Technion

Computation-Friendly Game Representations

- In practice, interesting games are large;
  computing equilibrium is hard

- CS agenda
  - compact representation
  - tractable computation

- Independence
  - some agents have no (direct) effect on each other's payoffs

    [La Mura, 2000], [Kearns, Littman, Singh, 2001], [Vickrey & Koller, 2002], [Oritz & Kearns, 2003], [Blum, Shelton, Koller, 2003]

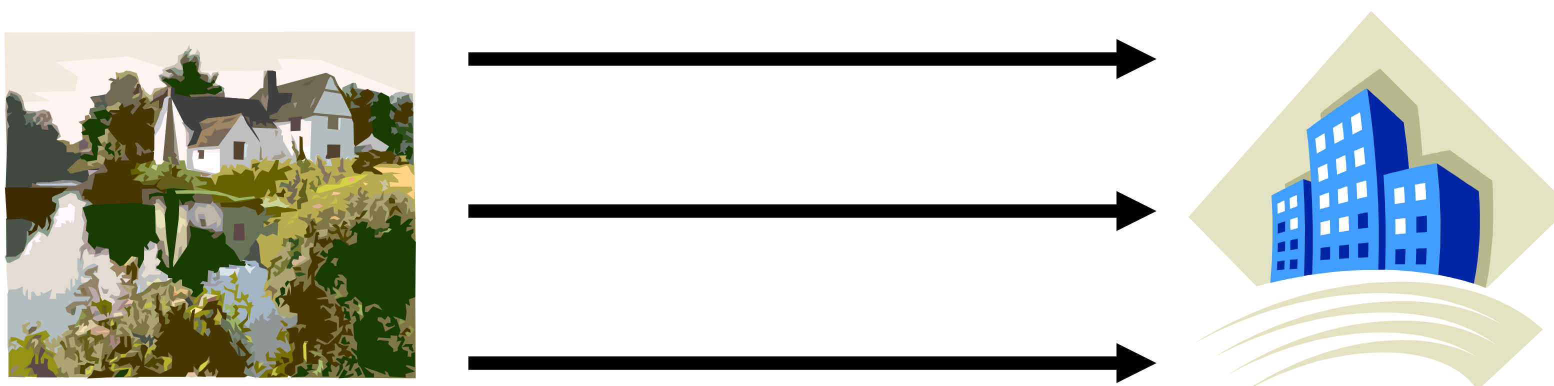
- Symmetry and Anonymity
  - all agents have the same utility function
  - agents affect each other in the same way

    [Roughgarden & Tardos, 2001], [Kearns & Mansour, 2002], [Rosenthal, 1973]

Congestion Games: Example

- Simplified congestion games: one resource per action
  - $D(a)$ is the number of agents who choose action a
  - $F_a(\cdot)$ are arbitrary functions for each a
  - agent i's utility: $\boldsymbol{u_i(a_i, D) = F_{a_i}(D(a_i))}$

- Congestion game example: traffic congestion



suburb *highways* *city*

Congestion and Potential Games

- **Congestion games** [Rosenthal, 1973]
  - set of resources R, actions A, each action $a \in A$ is a subset of R
    - agent i's action choices come from $A_i \subseteq A$
  - $D(r)$ is the number of agents who chose actions $a \mid r \in a$
  - $F_r(\cdot)$ are arbitrary functions for each $r \in R$
  - agent i's utility: $\boldsymbol{u_i(a, D) = \sum_{r \in a} F_r(D(r))}$
  - especially interesting: always have pure strategy Nash equilibria

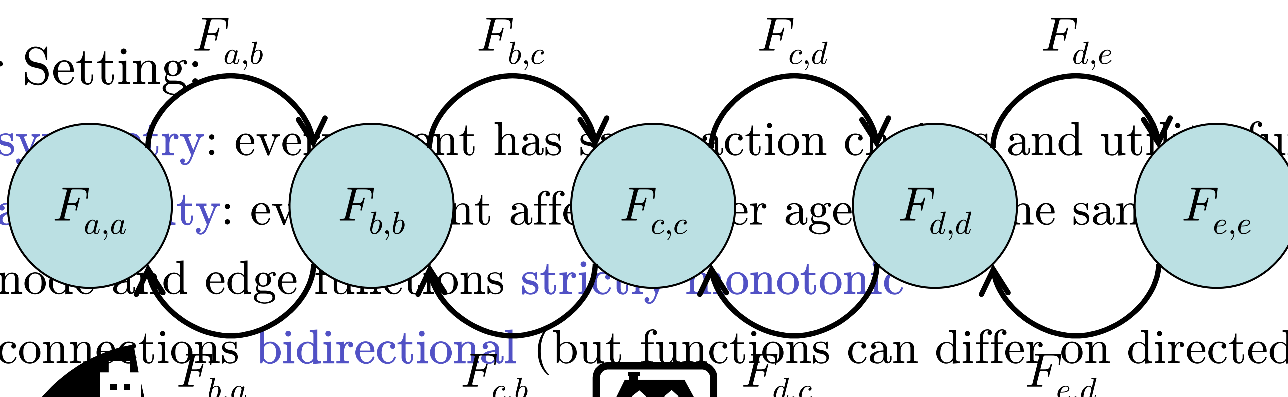
- **Potential games** [Monderer & Shapley, 1996]
  - let X and Y be tuples of agents' action choices, differing only in the choice of agent i
  - there exists a function P where $P(X) - P(Y) = u_i(X) - u_i(Y)$
  - equivalent to congestion games

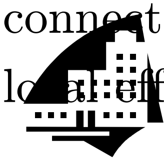
Local-Effect Games

- Sometimes, an agent is made to pay more because another agent chooses a different but related action
  - location problem: ice cream vendors on the beach
  - role formation game: choose skill in which to specialize

- Express relationships between actions with a local-effect graph
  - a node for every action a, labeled with a node function $F_{a,a}(\cdot)$
  - a directed edge from action a to action a' if a affects a', labeled $F_{a,a'}(\cdot)$
    - $neigh(a)$ is the set of actions that locally affect agents who choose action a

- Our Setting:
  - symmetry: every agent has same action choices and utility function
  - anonymity: every agent affects other agents the same way
  - node and edge functions strictly monotonic
  - connections bidirectional (but functions can differ on directed edges)
  - local effects additive





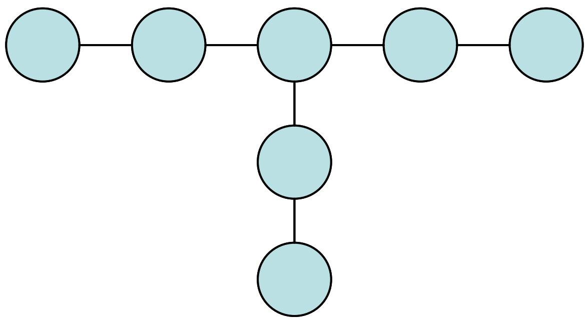
- Utility: $u_i(a, D) = F_{a,a}(D(a))$

Overview

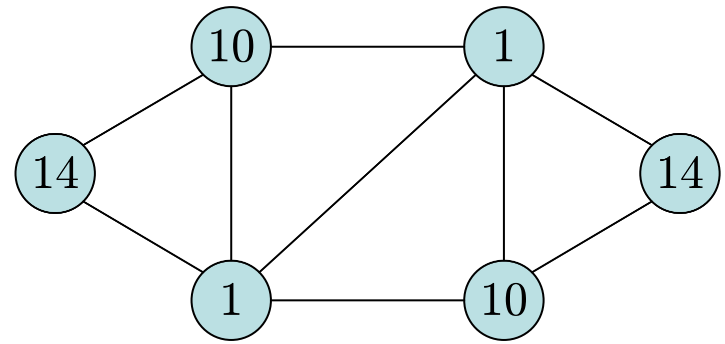
1. LEGs are a new game representation
   - compact: symmetry, anonymity, additivity, context-specific independence between actions
   - are LEGs different from (unsimplified) congestion and potential games?
     - we characterize the intersection between the classes of games

2. What about finding equilibria?
   - Computational experiments
     - myopic best-response dynamics
   - Theoretical – cases where:
     - LEGs can be reduced to potential games
     - no reduction to potential games, but PSNE still exist
     - no PSNE exist at all

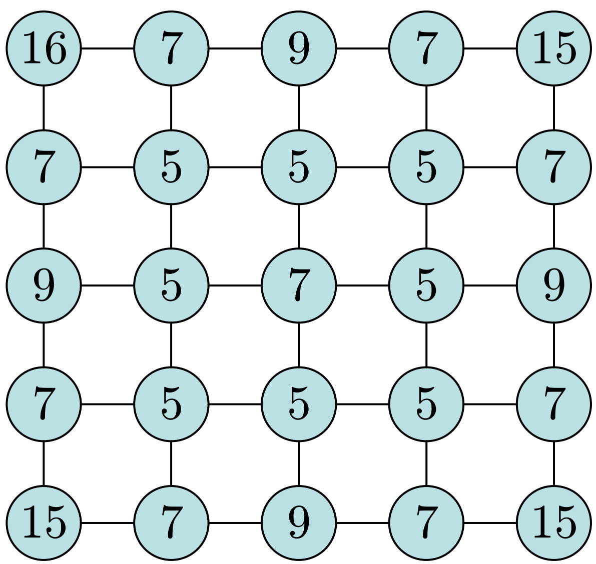
Computational Results



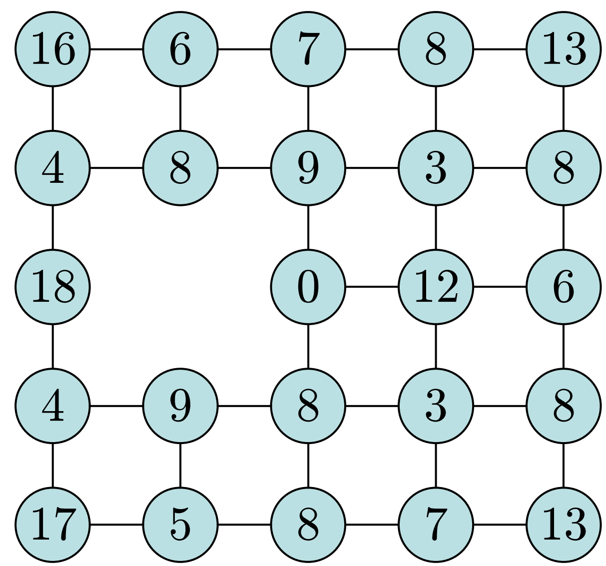
node: $3 \cdot \log(x+1)$ edge: $\log(x+1)$
50 agents



log/log. Node 3, edge 1. 50 agents



log/log; node 4, edge 1; 200 agents



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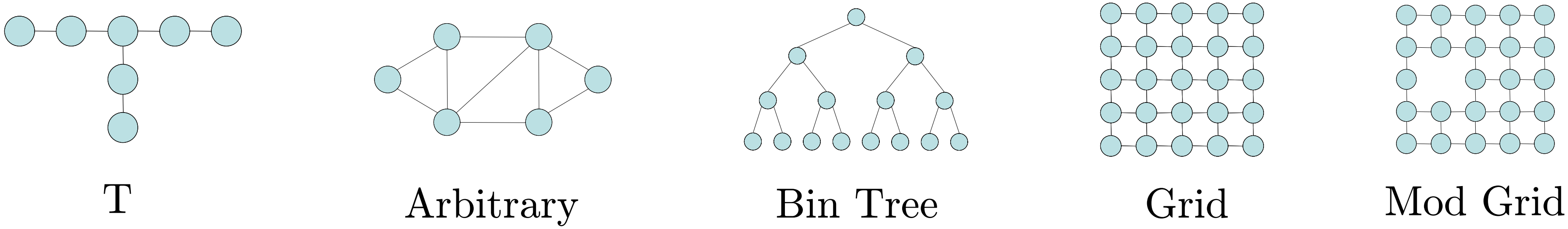
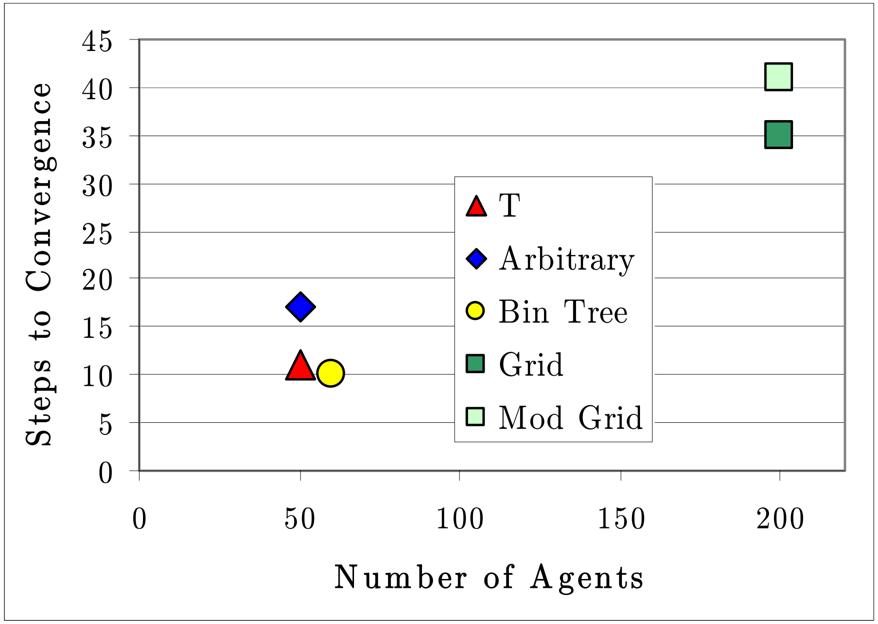
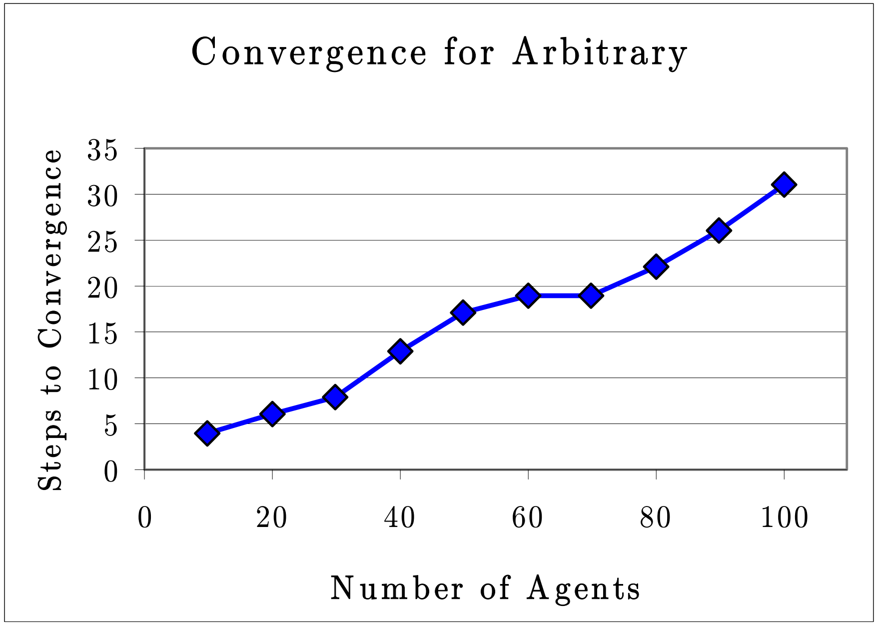


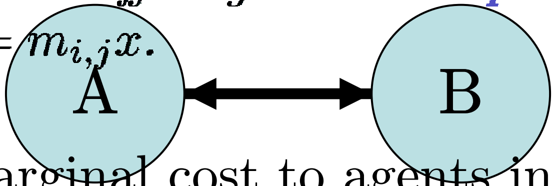




Bidirectional Local-Effect Games

Definition 1 *A local-effect game is a* ***bidirectional local-effect game*** *when* $\forall a \in \mathcal{A}, \forall a' \neq a \in \mathcal{A}$, $\mathcal{F}_{a,a'}(x) = \mathcal{F}_{a',a}(x)$.

Theorem 1 *Bidirectional local-effect games have* ***pure strategy Nash equilibria*** *if* $\forall i, \forall j \neq i\ \mathcal{F}_{i,j}(x) = m_{i,j}x$.



Linear edge functions: marginal cost to agents in A when one additional agent chooses B does not depend on total number of agents choosing B

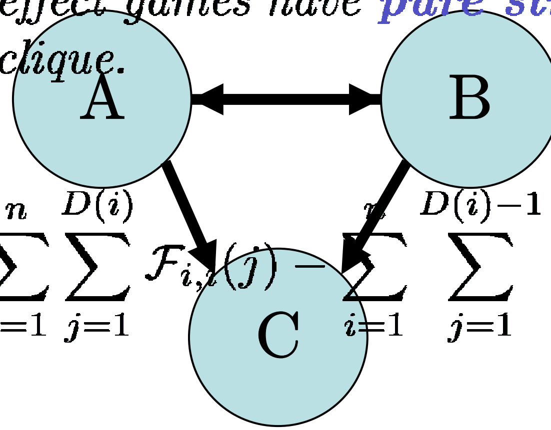
Proof by construction of a potential function:

$$P(D) = \sum_{i=1}^{n} \sum_{j=1}^{D(i)} \mathcal{F}_{i,i}(j) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j \in neigh(i)} D(i) m_{j,i} D(j)$$

Uniform Local-Effect Games

Definition 2 *A local-effect game is a* ***uniform local-effect game*** *when* $\forall A, B, C \in \mathcal{A}\ (B \in neigh(A) \wedge C \in neigh(A)) \rightarrow \forall x\, \mathcal{F}_{A,B}(x) = \mathcal{F}_{A,C}(x)$.

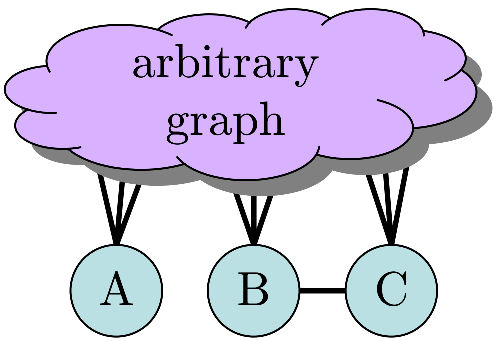
Theorem 2 *Uniform local-effect games have* ***pure strategy Nash equilibria*** *if the local-effect graph is a clique.*



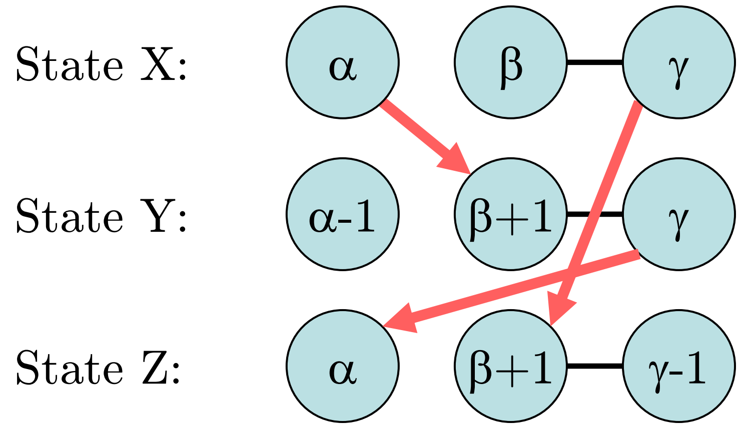
$$P(D) = \sum_{i=1}^{n} \sum_{j=1}^{D(i)} \mathcal{F}_{i,i}(j) - \sum_{i=1}^{n} \sum_{j=1}^{D(i)-1} \mathcal{F}_i^u(j)$$

LEGs and Potential Functions

Lemma 1 *The class of potential games does not contain the class of local-effect games when* $\exists A, B, C \in \mathcal{A}$ *where* $B \in neigh(C)$ *and not* $A \in neigh(B)$ *and not* $A \in neigh(C)$ *and* ($\mathcal{F}_{B,C} \neq \mathcal{F}_{C,B}$ *or* $\mathcal{F}_{B,C}$ *is nonlinear).*



(arbitrary node, edge functions)



Assume the existence of a potential function P.

$$P(X) - P(Y) = [u_A(\alpha)] - [\mathcal{F}_{C,B}(\gamma) + u_B(\beta+1)] \tag{1}$$

$$P(X) - P(Z) = [\mathcal{F}_{B,C}(\beta) + u_C(\gamma)] - [\mathcal{F}_{C,B}(\gamma-1) + u_B(\beta+1)] \tag{2}$$

$$P(Y) - P(Z) = [\mathcal{F}_{B,C}(\beta+1) + u_C(\gamma)] - [u_A(\alpha)] \tag{3}$$

$$\begin{aligned} P(Y) - P(Z) &= [P(X) - P(Z)] - [P(X) - P(Y)] \\ &= \big[[\mathcal{F}_{B,C}(\beta) + u_C(\gamma)] - [\mathcal{F}_{C,B}(\gamma-1) + u_B(\beta+1)]\big] - \big[[u_A(\alpha)] - [\mathcal{F}_{C,B}(\gamma) + u_B(\beta+1)]\big] \end{aligned} \tag{4}$$

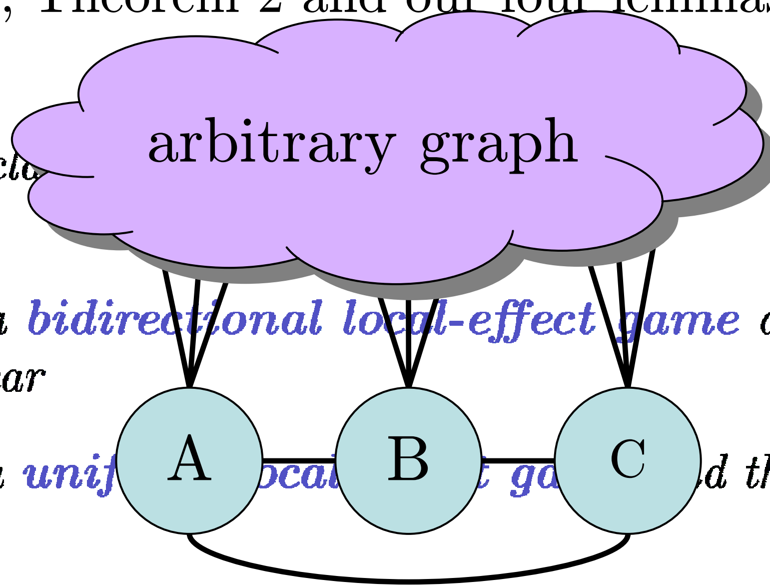
Intersect equations (3) and (4): $\quad \mathcal{F}_{C,B}(\gamma) - \mathcal{F}_{C,B}(\gamma-1) = \mathcal{F}_{B,C}(\beta+1) - \mathcal{F}_{B,C}(\beta)$.

It must be that $\mathcal{F}_{B,C} = \mathcal{F}_{C,B}$ and $\mathcal{F}_{B,C}$ is linear: a contradiction.

Potential Games

- Three other lemmas:
  - subgraphs of three nodes with other connectivities
  - graphs having fewer than three nodes

- Using Theorem 1, Theorem 2 and our four lemmas, we prove:



Theorem 3 *The cl… ass of local-effect games when:*

1. *the game is a **bidirectional local-effect game** and all local-effect functions are linear*
2. *the game is a **unif** … **ocal** … **t ga** … d the local-effect graph is a clique*

No other class of local-effect games belongs to the class of potential games.

Other Cases

- We can find pure strategy Nash equilibria even in cases where no potential function exists

- Theorem 4 *When*
  - *node effect functions dominate edge effect functions*
  - *edge effect functions are sublinear*

  *then there **exists a PSNE** in which agents choose nodes that constitute an independent set*

- There are LEGs for which no PSNE exists
  - verified by exhaustive enumeration of pure strategies

Conclusions

- Local-Effect Games offer a novel compact representation
  - exploiting symmetry, anonymity, additivity and context-specific independence in utility functions
  - very natural graphical representation
- LEGs not equivalent to potential/congestion games
  - we characterized exactly which LEGs *are* potential games
- Many LEGs have pure-strategy Nash equilibria
  - three subclasses shown theoretically
  - however, PSNE do not always exist
- Even when LEGs cannot be proven to have PSNE, equilibria can often be found experimentally using myopic best-response dynamics

http://robotics.stanford.edu/~kevinlb google://"Kevin Leyton-Brown"

Thanks for your attention!

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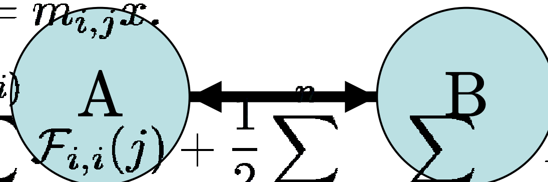

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- First sum: congestion game cost function (summing over agents)
  - we know every congestion game has a potential function
  - since PFs are additive, we can use this function to explain node functions if we can find another term to capture the effect of edge functions

- Global utility change due to local effects when agent deviates from D to D' is:

$$\sum_{i=1}^{n} \sum_{j \in neigh(i)} D(i) m_{j,i} D(j) - \sum_{i=1}^{n} \sum_{j \in neigh(i)} D'(i) m_{j,i} D'(j)$$

  - from linearity & bidirectionality, aggregate utility change imposed on all other agents is the same as the utility change imposed on self

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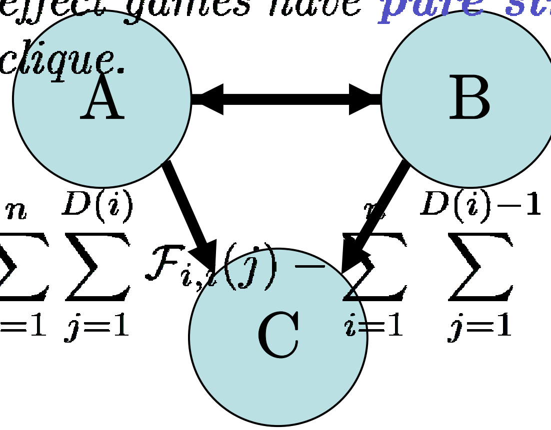

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- The graph is a clique
  - the only node that does *not* locally affect an agent is the node corresponding to her action
- Consider $P(D)$ - $P(D')$, where D, D' differ for only one agent:
  - the only terms that do not cancel out from the second summation are the local effect from the original action and the new action